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Faster Impedance Estimation for Coupled Microstrips with an Overrelaxation Method

R. DAUMAS, D. POMPEI, E. RIVIER, AND A. ROS

***Abstract*—Using the Frankel–Young method [1], [2], fast estimation of the potential distribution for a microstrip structure is obtained when an accelerating factor ω is introduced in the finite-differences (relaxation) method. It is possible to calculate such a factor by an iterative technique, but the time of computation needed to find ω annihilates the theoretical gain.**

In this short paper, the authors present a method which gives an analytical expression for ω. The realistic case examined here, as an illustration, is that of the suspended microstrip couplers for which odd and even impedances are the interesting parameters.

Given an analytical expression for ω, the overrelaxation method appears as a very powerful and attractive method for finding the solution of any type of microstrip structure.

I. Introduction

The integrated technology using microstrips provides new possibilities for microwave designs. A very important one is the realization of compact low-cost dispersive lines used as group-velocity correctors for digital telecommunications. The basic component of such a system can be reduced to a microstrip coupler.

In the last few years, several authors [3]–[7] have treated some particular problems using different methods, but they are generally complicated and applicable to particular geometrical cases.

A solution using finite differences has been proposed by Green [8] and others. An application has been given by Brenner [9] to the simple case of the suspended microstrip line and by Gupta [10] to the idealized problem considered by Cohn [3], i.e., the suspended coupler in a homogeneous dielectric such as air. That problem is purely theoretical, with no substrate sustaining the strips.

However, as emphasized by Smith [6], the methods using finite differences appear as inadequate because the very fine mesh required for the accuracy leads to difficulties in the convergence. Clearly, it means that the computing time becomes prohibitive and the computer memory becomes saturated.

Nevertheless, the use of the finite-differences method should become a very fruitful approach if an accelerating factor taking into account the geometry of the problem could be injected in the program.

In this short paper, the problem of the research of such a factor is solved and applied to the case of the suspended coupler with a dielectric substrate of constant ϵ_r sustaining the strips.

In the finite-differences method, we define in a geometrical domain the potentials at the nodes of a net (Fig. 1). The relations between all the potentials can be written

$$(A)(\Psi) = (B). \tag{1}$$

System (1) can be solved by an iterative process [1], [2], [11] writing

$$\Psi_{i+1} = M\Psi_i + C. \tag{2}$$

The Frankel–Young method introduces the accelerating factor ω. An optimal value of $\omega = \omega_{opt}$ gives the fastest convergence. We have

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \lambda_M{}^2}} \tag{3}$$

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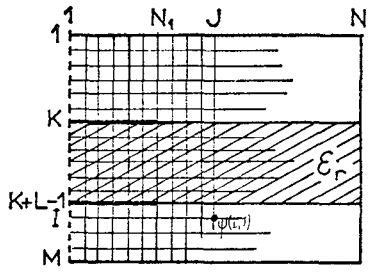

Fig. 1. Geometrical parameters definition.

$$\lambda_M = \sup \left| 1 - \frac{\mu_k}{a_{kk}} \right| \tag{4}$$

where μ_k are the eigenvalues and a_{kk} are the diagonal elements of A.

II. Applications of the Frankel–Young Method to a Suspended Microstrip Coupler

A. Resolution of the Problem for an "Empty Box" [11]

Using the finite-differences method and for the second-order approximation, the Laplace's equation is reduced to

$$\Psi(I, J) = \tfrac{1}{4}[\Psi(I, J-1) + \Psi(I, J+1) + \Psi(I-1, J) + \Psi(I+1, J)]. \tag{5}$$

The variable changing $\psi(I, J) = X_i$, with $i = (N-2)(I-2) + J - 1$, allows us to have the unknowns indexed by a continuous sequence (Fig. 1).

The ith equation of the system (1) will be written

$$4X_i - X_{i-1} - X_{i+1} - X_{i+N-2} - X_{i-N+2} = 0, \qquad 1 \le i \le (M-2)(N-2).$$

The matrix A_1 (Fig. 2) can always be split into two symmetrical tridiagonals—matrices A_1 and A_2, the main diagonal elements being $a_{kk}/2$ such that $A = A_1 + A_2$.

It can be shown that A_1 and A_2 have the same eigenvectors. Let V be one of these eigenvectors and μ_1 and μ_2 be the two corresponding eigenvalues for A_1 and A_2; thus we have

$$AV = (\mu_1 + \mu_2)V.$$

Consequently, the eigenvalue of A corresponding to V is

$$\mu = \mu_1 + \mu_2.$$

The matrix A_1 has $(M-2)$ tridiagonal blocks of order $(N-2)$, where all are identical; let A_1' be such a block. The eigenvalues μ_1 of A_1 are $(M-2)$ times the eigenvalues of A_1':

$$A_1' = \begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ . & . & . & & \\ & . & . & . & \\ & & . & . & . \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{vmatrix}_{N-2}$$

The eigenvalues of the matrix A_1' are

$$\mu_{1k}' = 2 - 2\cos\frac{k\Pi}{N-1}, \qquad 1 \le k \le N-2.$$

In the matrix A_2, by permutations of rows and columns, it is possible to reduce A_2 to a band matrix $\bar{A}_2$ like A_1.

In order to avoid the tedious calculations by permutations, it is possible to find a faster process to transform A_2 into $\bar{A}_2$. One chooses another variable changing $\psi(I, J) = X_i{}^*$, with $i = (M-2)(J-2) + I - 1$.

The system (1) is then written $(A)^*(X)^* = (B)^*$. The physical problem is unchanged, so that the solution is the same. The solution vector $(X)^*$ is simply written on a new set of coordinate vectors. A and A^* represent the same linear application; consequently, they are similar and therefore have the same eigenvalues.

This time, A^* can be split into two matrices $A_1{}^*$ and $A_2{}^*$, where $A_1{}^*$ originates from the vertical lines and $A_2{}^*$ from the horizontal lines and $A^* = A_1{}^* + A_2{}^*$.

$A_1{}^*$ has $(N-2)$ diagonal blocks $A_1'{}^*$ similar to A_1', but of order $M-2$.

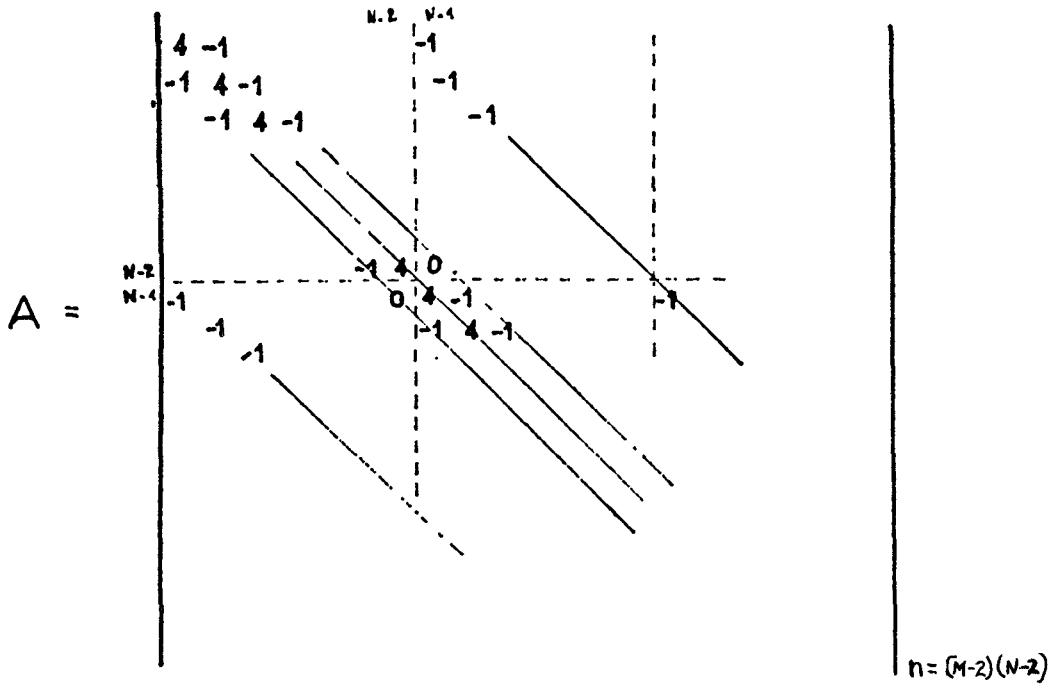

Fig. 2. Matrix A for an empty box.

Here, we can also say that μ^*, the eigenvalue of A^* corresponding to V, can be written $\mu^* = \mu_1^* + \mu_2^*$, where μ_1^* and μ_2^* are the eigenvalues corresponding to the eigenvectors for the matrices A_1^* and A_2^*.

μ_1^* is one of the values $2-2\cos(l\Pi)/(M-1)$, $1<l\leq M-2$, depending only on M characterizing the vertical dimension of the box, since A_1^* originates from the vertical lines of the rectangular box.

A and A^* having the same eigenvalues, we necessarily have $\mu=\mu^*$ or $\mu_1+\mu_2=\mu_1^*+\mu_2^*$; μ_2 and μ_1^* depend only on M; μ_1 and μ_2 depend only on N. Then this relation is verified whatever M and N are if $\mu_2=\mu_1^*$ and $u_1=\mu_2^*$.

So for finding μ we shall use either the writing A or A^* of the matrix, accordingly, as it is easier to calculate the eigenvalues of A_1 and A_2 or of A_1^* and A_2^*.

From (4), we have

$$\lambda_k = \frac{\cos\dfrac{k\Pi}{N-1} + \cos\dfrac{l\Pi}{M-1}}{2}$$

from which we have the inquired value for λ_{max}:

$$\lambda_M = \frac{\cos\dfrac{\Pi}{N-1} + \cos\dfrac{\Pi}{M-1}}{2}.$$

B. Suspended Microstrip Coupler

Let us introduce in the initial box a thin plate of dielectric of permittivity ϵ_r. This plate supports two strips of negligible thickness (Fig. 1).

The various coefficients of the matrix A are modified. The calculations of the eigenvalues of A would be made by the classical methods, using iterative processes (radius of convergence, Rayleigh quotient, etc. [12], [13]). Meanwhile, these methods are complicated and as soon as M and N are of the order of some tens, the time of calculation becomes extensive. Moreover, they are convenient only if the eigenvalues are not degenerate or 2×2 opposite, which is just the case here. Therefore, we have looked at a method allowing us to directly give a very good approximation of η_M by trying to calculate explicitly the eigenvalues of matrix A.

This can be made by noticing that the introduction of the strip has an effect in the system (1) only on the relations between the terms of a same row, whereas the introduction of the dielectric modifies only the coefficients of the terms of a same column, for the crossing of the two air–dielectric interfaces.

The symmetrization of the problem introduces, like supplementary unknowns, the potentials of the points corresponding to $J=1$ (Fig. 1). Then each horizontal line has $(N-1)$ nodes. We have $(M-2)(N-1)$ unknowns. Hence, A_1 consists of $(M-2)$ blocks like A_1':

$$A_1' = \begin{vmatrix} 2 & -2 & & & & \\ -1 & 2 & -1 & & & \\ . & . & . & & & \\ & . & . & . & & \\ & & & . & . & . \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & \end{vmatrix}_{N-1}$$

$$A_1'' = \begin{vmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ . & . & . & & & \\ & . & . & . & & \\ & & & . & . & . \\ & & & & & -1 \\ & & & -1 & 2 & \end{vmatrix}_{(N-1)-N_1}$$

Effect of the Strip: The introduction of the strip, in the case of a coupler, leaves out $2\times N_1$ unknowns, because the potentials on the conductors are fixed to $+1$ in the case of the even mode and to ±1 in the case of the odd mode.

In A (Fig. 3), we substitute $2\times N_1$ elements of the main diagonal by 1, the other elements of the corresponding rows all being nulls. These terms are those of the N_1 first rows of the Kth and $(K+L-1)$th blocks of the matrices A_1 and A_2.

The elements a_{kk} of A, equal to 1, give, after the expansion of the determinant $\|A-\mu U\|$, with respect to rows having only terms equal to 1, $2\times N_1$ times the eigenvalue $\mu=1$. According to relation (4), this corresponds to $2\times N_1$ times the eigenvalue $\lambda=0$ for the matrix M. As we research λ_{max}, these eigenvalues can be excluded, and also the corresponding diagonal terms of A.

Then, all the other eigenvalues of A_1 will be $(M-4)$ times the eigenvalues of A_1', plus two times the eigenvalues of A_1''.

We obtain, for μ_{1k}', eigenvalues of A_1':

$$\mu_{1k}' = 2 - 2\cos\frac{\Pi + 2k\Pi}{2(N-1)}, \qquad 0 \leq k \leq N-2.$$

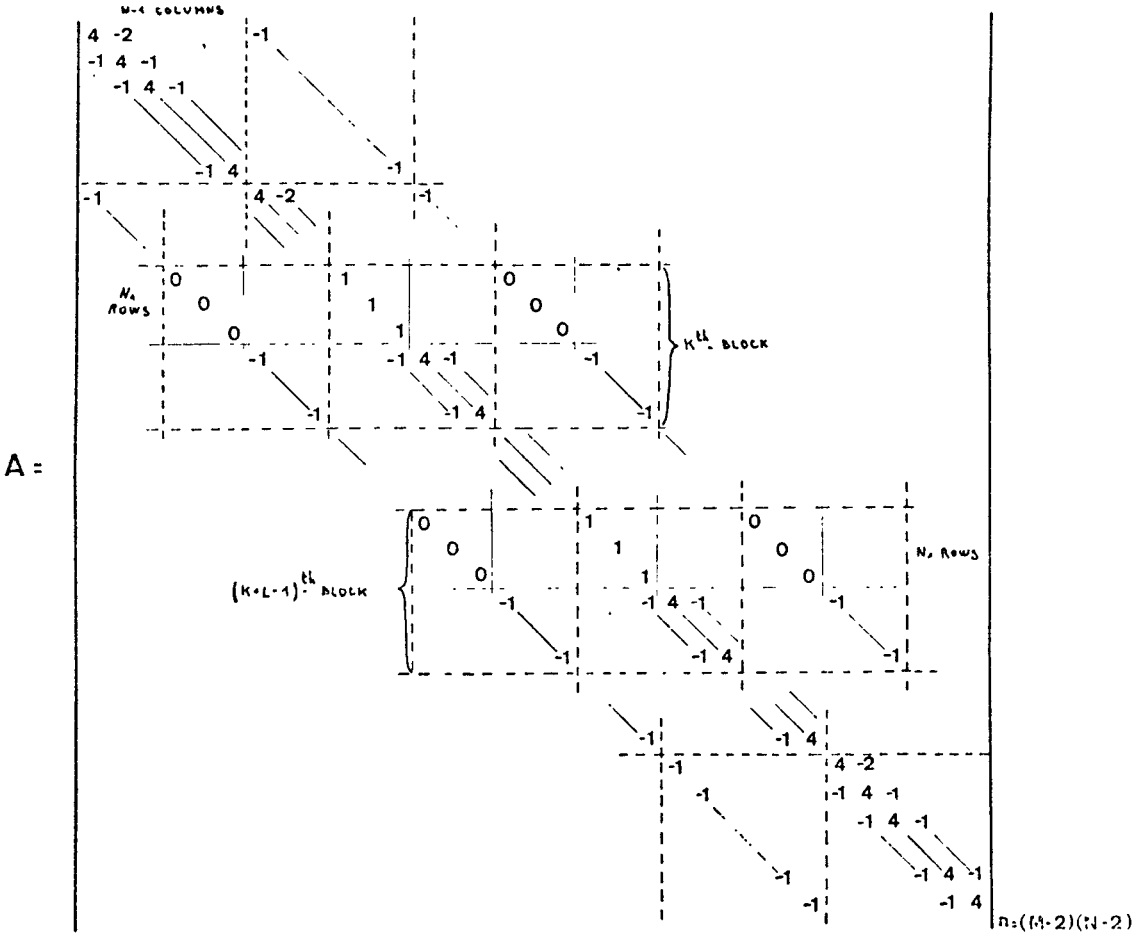

Fig. 3. Matrix A for a suspended microstrip coupler.

We require the smallest possible value of μ_{1k}' to make λ the highest one (4). Thus

$$\mu_{1\min}' = 2 - 2\cos\frac{\Pi}{2N-2}.$$

The eigenvalues of A_1'' are

$$\mu_1'' = 2 - 2\cos\frac{k\Pi}{N-N_1}, \qquad 1 \le k \le N - N_1 - 1 \quad \text{and}$$

$$\mu_{1\min}'' = 2 - 2\cos\frac{\Pi}{N-N_1}.$$

So we take

$$\mu_{1\min} = 2 - 2\cos\frac{\Pi}{2N-2}$$

where $2N-2$ is always higher than $N-N_1$.

Effect of the Dielectric: The introduction of the dielectric modifies the relations between the potentials of adjacent nodes [11], but only the coefficients of the terms flanking the central point on the same vertical lines are modified. The matrix A_1 stays unchanged. Only A_2 will have some elements, depending on the relative permittivity of dielectric ϵ_r.

Here, we use the representation A_1^*, instead of A_2; A_1^* has $N-1$ blocks of order $M-2$. After elimination of rows containing only 1 on the main diagonal, the eigenvalues of A_1^* will be N_1 times the eigenvalues of

$$\begin{vmatrix} A_2' & & \\ & A_2'' & \\ & & A_2''' \end{vmatrix}$$

see Fig. 4, and $(N-1)-N_1$ times the eigenvalues of A_2'''' (Fig. 5).

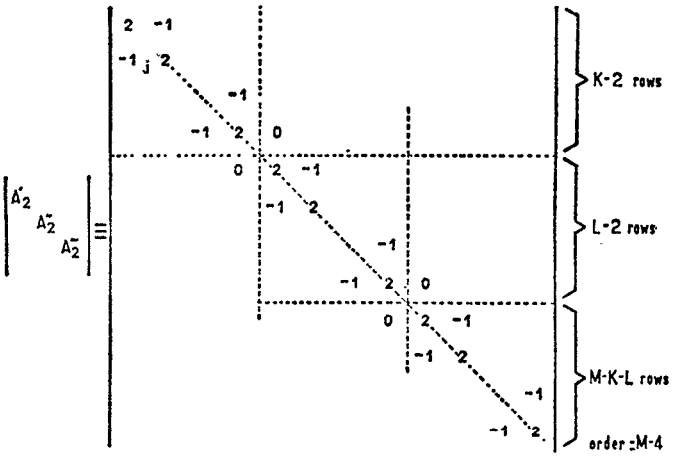

Fig. 4.

If

$$\phi = \inf\left(\frac{\Pi}{K-1}, \frac{\Pi}{L-1}, \frac{\Pi}{M-K-L-1}\right)$$

the smallest eigenvalue of A_2', A_2'', and A_2''' is $\mu_{2m}' = 2 - 2\cos\phi$. A_2'''' gives, for the smallest eigenvalue,

$$\mu_{2\min}'' = 2 - 2\cos\phi'$$

where ϕ' is the first nonnull root of the equation

$$F(\phi') = \sin(M-1)\phi' - \alpha^2\sin(M-2L+1)\phi' + \alpha[\sin(M-2K+1)\phi' - \sin(M-2K-2L+3)\phi'] = 0$$

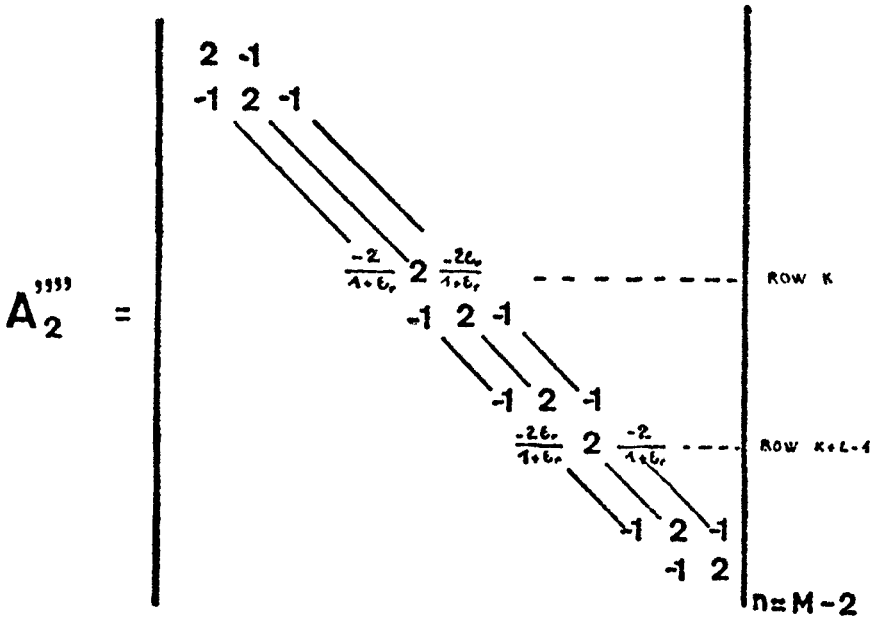

Fig. 5.

TABLE I

TEST$_1 = 10^{-3}$

	ω	Iteration Number $= IT$	Precision	ϵ_r
Empty box	1.79148	45	$2.5 \cdot 10^{-2}$	1
Corrected value	1.856725	36	$1.25 \cdot 10^{-4}$	1

Gain for the iteration number—20 percent.

Gain for the precision—ratio 20.

Remarks about this corrected value of ω—Choosing values separated by about one part in a thousand and for the same number of iterations, the precision is 10–20 times below.

TABLE II

TEST$_1 = 10^{-3}$; EVEN MODE

	ω	Iteration Number $= IT$	Precision	ϵ_r
ω_0 (ω calculated for the empty box)	1.79148	31	$2.7 \cdot 10^{-2}$	1
ω_1 (ω strip corrected)	1.856725	32	$3.25 \cdot 10^{-3}$	1
ω_2 (ω strip and dielectric corrected)	1.856725	32	$3.25 \cdot 10^{-3}$	1
ω_0	1.79148	30	$2.6 \cdot 10^{-2}$	2
ω_1	1.856725	32	$3 \cdot 10^{-3}$	2
ω_2	1.8651	34	$1.5 \cdot 10^{-3}$	2
ω_0	1.79148	28	$2.88 \cdot 10^{-2}$	4
ω_1	1.856725	32	$2 \cdot 10^{-3}$	4
ω_2	1.876531	36	$4.3 \cdot 10^{-4}$	4
ω_0	1.79148	27	$2.64 \cdot 10^{-2}$	8
ω_1	1.856725	29	$6.8 \cdot 10^{-3}$	8
ω_2	1.886921	43	$2.86 \cdot 10^{-3}$	8

TABLE III

TEST$_1 = 10^{-3}$; EVEN MODE—$\epsilon_r = 4$

ω	$\omega_0 = 1.79648$	$\omega_1 = 1.856725$	$\omega_2 = 1.876531$
Precision	$9.4 \cdot 10^{-3}$	$1.56 \cdot 10^{-3}$	$4.3 \cdot 10^{-4}$

Gain for precision = factor 22.

TABLE IV

EVEN MODE—$\epsilon_r = 4$

	Test$_2 = 10^{-2}$		Test$_2 = 10^{-3}$	
ω	IT	Precision	IT	Precision
ω_0	36	$9.42 \cdot 10^{-3}$	52	$8.85 \cdot 10^{-4}$
ω_1	17	$6.46 \cdot 10^{-3}$	39	$7.76 \cdot 10^{-4}$
ω_2	15	$2.4 \cdot 10^{-3}$	36	$4.3 \cdot 10^{-4}$
Gain for the iteration number	58 percent		31 percent	

TABLE V

EVEN MODE—$\epsilon_r = 4$

Precision	$2.4 \cdot 10^{-3}$	$4.3 \cdot 10^{-4}$	
Iteration number	515	660	without accelerating factor
	15	36	with ω_{opt}

with

$$\alpha = 1 - \frac{2\epsilon_r}{1 + \epsilon_r} = \frac{1 - \epsilon_r}{1 + \epsilon_r}.$$

So $\mu_{2\min} = 2 - 2 \cos \theta$, with $\theta = \inf (\phi \cdot \phi')$.

Total Effect—Strip and Dielectric: Then we make the supplementary assumption that the eigenvectors of A_1 and A_2 remain the same, as in the case of an empty box. This will be a good approximation, as long as the number of disturbed nodes will be small before the total number of nodes.

Under these conditions, we can write, for the whole system studied,

$$\mu_{\min} = \mu_{1\min} + \mu_{2\min} \quad \text{and} \quad \lambda_{\max} = \frac{1}{2}\left[\cos \frac{\Pi}{2N - 2} + \cos \theta\right].$$

III. Numerical Results

The effect of the new value of ω_{opt} deduced by all that precedes has been tested for some configurations. As an example, the results for $M = 39$, $N = 23$, $N_1 = 5$, $K = 10$, and $L = 9$ are given in Tables I–V.

For this configuration and $\epsilon_r = 4$, we have obtained

$$Z_{0o} = 47.75\,\Omega, \qquad Z_{0e} = 127.2\,\Omega.$$

To have an idea of the precision of the potential calculation, we have calculated the capacitance per unit length of the stripline by application of Gauss' theorem for two surfaces. The first, near the strip, gives C_I, and the second, near the external walls of the box, gives C_0. We call relative precision of the calculation the ratio $[C_I - C_0]/C_I$.

Table I gives data for a simple line when we stop the calculation if the highest difference between the potentials of the corresponding nodes for two successive iterations is smaller than a fixed value, called "test$_1$."

Table II gives data for a microstrip coupler. Here, C_I is the capacitance calculated for a surface surrounding one of the two strips, for instance, the positively charged one in the odd mode. C_0 is calculated for a surface containing all the other conductors.

Table III gives results for the same number of iterations (36) in the case of the coupler with $\epsilon_r = 4$.

Conversely, in Table IV, we give the results when we stop the computation if the "relative precision" is smaller than a fixed value called "test$_2$."

It should be noted that for very small differences (a few percent) in the value of ω, the number of iterations and the precision are perceptibly different. This effect has been observed and justified by some authors [20]–[24].

In Table V, as an illustration of our method against the Gauss–Seidel one, we give the data as for Table IV, with the best accelerating factor and without the accelerating factor.

IV. Conclusion

From these results, it can be seen that the approximations made for the calculation of an accelerating factor are very good. In each case we have tested, we have obtained an important amelioration either in the computational time or in precision of calculation, often for both.

The precision of the finite-differences method is sufficiently good, as can be seen by comparison with results given by others.

We have used two types of results. The first ones are those obtained by Cohn's formulas [11]. In our program, making $\epsilon_r = 1$ and $N \gg M$, L, N_1, we must approach Cohn's case.

For example, we have obtained the following:

M	N	K	L	N_1	Z_{0o} (calculated)	Z_{0e} (calculated)	Z_{0o} (Cohn)	Z_{0e} (Cohn)
21	101	9	5	6	43.3	143.7	45.6	148.8
21	101	9	5	16	17.7	73.6	18.1	74.8
51	101	24	5	6	46.5	239.2	47.9	249.3
51	101	24	5	16	19.7	153	20.1	158.2
53	132	24	7	18	24.7	137.5	25	143.8

We have made 17 comparisons between our results and those calculated using Cohn's formula. The mean accuracy of these results is about 2 percent.

Second, experimental results have been obtained by the Centre National d'Etudes des Télécommunications, Lannion, France. Comparison with these results gives an accuracy of 4 or 5 percent. For example, we have the following:

M	N	K	L	N_1	Z_{0o} (calculated)	Z_{0e} (calculated)	Z_{0o} (experimental)	Z_{0e} (experimental)
18	44	9	2	6	16.15		16.17	
72	88	35	4	19	11	115	10.05	111.3

The only point which can be noted in opposition to this method is that for realistic problems, a computer of great capacity is necessary.

Finally, the finite-differences method appears in many aspects to be the most simple to use for the calculations of microstrip parameters in the TEM approximation.

This work allows a reduction of the computational time necessary in the finite-differences method using the SOR technique of 20–60 percent according to the desired accuracy.

Acknowledgment

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On Uniform Multimode Transmission Lines

CLAYTON R. PAUL

***Abstract*—In a recent short paper [1], a method for constructing solutions to the classical uniform multiwire transmission-line equations was given which was intended to include the case of partial eigenvalue degeneracy. This development appears to be incorrect and a correct development will be given. In addition, a complete method for constructing the matrix chain parameters of a section of line will be presented.**

We will consider n uniform transmission lines described by the matrix partial differential equations

$$\frac{\partial \boldsymbol{v}(x,t)}{\partial x} = -\boldsymbol{R}\boldsymbol{i}(x,t) - \boldsymbol{L}\frac{\partial \boldsymbol{i}(x,t)}{\partial t} \tag{1a}$$

$$\frac{\partial \boldsymbol{i}(x,t)}{\partial x} = -\boldsymbol{G}\boldsymbol{v}(x,t) - \boldsymbol{C}\frac{\partial \boldsymbol{v}(x,t)}{\partial t} \tag{1b}$$

where $\boldsymbol{v}(x,t)$ and $\boldsymbol{i}(x,t)$ are $n \times 1$ vector functions of the transmission-line voltages with respect to some reference conductor (usually a ground plane) and currents, respectively, as a function of distance x along the line and time t. The matrices $\boldsymbol{R}$, $\boldsymbol{L}$, $\boldsymbol{G}$, and $\boldsymbol{C}$ are $n \times n$ matrices independent of x. Nonuniform transmission lines would have $\boldsymbol{R}$, $\boldsymbol{G}$, $\boldsymbol{L}$, and $\boldsymbol{C}$ as functions of x. Usually, $\boldsymbol{R}$ is diagonal and $\boldsymbol{G}$, $\boldsymbol{L}$, and $\boldsymbol{C}$ are symmetric (for lines emersed in linear, isotropic media). By invoking the Laplace transform with respect to time, we arrive at the equations

$$\frac{d\boldsymbol{V}(x)}{dx} = -\boldsymbol{Z}\boldsymbol{I}(x) \tag{2a}$$

$$\frac{d\boldsymbol{I}(x)}{dx} = -\boldsymbol{Y}\boldsymbol{V}(x) \tag{2b}$$

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